STA 247 — Solutions to Assignment #1

Question 1: Suppose you throw three six-sided dice (coloured red, green, and blue) repeatedly, until the three dice all show different numbers. Assuming that these dice are equally likely to show any combination of three different numbers, calculate the probabilities of the following events:

a) The green die shows the number 6.

The sample space for this problem consists of all combinations of values for the red, green, and blue dice in which the numbers are all distinct. There are $6 \times 5 \times 4 = 120$ *such combinations, all of which are equally likely. (This can be seen by imagining that we assign one of the 6 numbers to the red die, then one of the remaining 5 numbers to the green die, then one of the remaining 4 numbers to the blue die.) The number of combinations in which the green die shows 6 is* $5 \times 4 = 20$ *(five possible value for the red die that aren't six, and then four possible values for the blue die that aren't six and aren't the same as the red die). The probability that the green die shows 6 is therefore* $5 \times 4 \ / \ 6 \times 5 \times 4 = 1/6$.

b) The sum of the numbers shown by the three dice is 6.

The only set of three distinct integers from 1 to 6 that add up to 6 is $\{1,2,3\}$. *These can be ordered in* $3! = 6$ *ways as values for the red, green, and blue dice. The probability of the three dice having values adding to 6 is therefore* $6/120 = 1/20$.

c) The sum of the numbers shown by the three dice is at least 14.

There are two sets of three distinct integers from 1 to 6 that add up to at least 14: $\{3,5,6\}$ *and* $\{4,5,6\}$. *Each of these can be ordered in 6 ways, giving a total of 12 ways that the three dice can add up to 14 or more. The probability of this is therefore* $12/120 = 1/10$.

d) All three dice show odd numbers.

There are three odd integers between 1 and 6, which can be arranged in $3! = 6$ *ways to give values for the red, green, and blue dice. The probability that the three dice show odd numbers is therefore* $6/120 = 1/20$.

Question 2: The conditional probability $P(C|A)$ represents how likely the event C is if we already know that the event A has occurred (and we know nothing else). If $P(C|A) > P(C)$, the occurrence of A increases our belief that C has occurred. If $P(C|B) > P(C)$ as well, we might expect that if we know that **both** A and B have occurred, our belief that C has occurred would be at least as great as when we know only that A occurred or only that only B has occurred. This intuition leads to the following conjecture:

Conjecture: Let A, B, and C be any three events, with $P(A) > 0$, $P(B) > 0$, and $P(A \cap B) > 0$. If $P(C|A) > P(C)$ and $P(C|B) > P(C)$, then $P(C|A \cap B) > P(C)$.

EITHER prove that this conjecture is true, using the three basic axiom of probability (p. 8 in the text) and the definition of conditional probability (p. 27), **OR** show that the conjecture is not true, by finding a specific example of a sample space, S, probabilities for outcomes in S, and events A, B, and C for which the conjecture is false.

We can see that the conjecture is false from the following example. Let the sample space be $S = \{1, 2, 3, 4, 5\}$*, with the five outcomes being equally likely. Let* $A = \{1, 2\}$*,* $B = \{1, 3\}$*, and* $C = \{2, 3\}$*. We can see that* $P(A) = 2/5$*,* $P(B) = 2/5$*, and* $P(A \cap B) = 1/5$ *are all greater than zero.* $P(C) = 2/5$ *while* $P(C|A) = 1/2$ *and* $P(C|B) = 1/2$*, so* $P(C|A) > P(C)$ *and* $P(C|B) > P(C)$*. But* $P(C|A \cap B) = 0$*, since* $A \cap B = \{1\}$ *and* $A \cap B \cap C = \emptyset$*. This is contrary to the conjecture's claim that* $P(C|A \cap B)$ *will be greater than* $P(C)$*.*

Question 3: You flip a coin. If the coin lands heads, you roll two six-sided dice. If the coin lands tails, you roll one six-side die. Define the following random variables:

X = 1 if the coin lands tails, 2 if the coin lands heads
Y = The sum of the numbers on all the dice rolled (either one or two dice)

Assuming that the coin and dice are fair,

a) Write down a table showing the joint probability mass function for X and Y.

We can use the multiplication rule, $P(X = x, Y = y) = P(X = x)\, P(Y = y|X = x)$*, to fill in the table. We know that* $P(X = 1) = P(X = 2) = 1/2$*.* $P(Y = y|X = 1)$ *is 1/6 for* $y \in \{1, 2, 3, 4, 5, 6\}$ *and zero otherwise, since when* $X = 1$*, we roll just one die. When we roll two dice, we have to consider the possible combinations when finding* $P(Y = y|X = 2)$*. For instance,* $P(Y = 4|X = 2) = 3/36$*, since we can get a total of 4 as* $1 + 3$ *or* $2 + 2$ *or* $3 + 1$*. The final table is as follows:*

		Y =										
	1	2	3	4	5	6	7	8	9	10	11	12
X = 1	1/12	1/12	1/12	1/12	1/12	1/12	0	0	0	0	0	0
2	0	1/72	2/72	3/72	4/72	5/72	6/72	5/72	4/72	3/72	2/72	1/72

b) Write down a table showing the marginal probability mass function for Y.

We just add the two numbers in each column of the table above, with the result being

Y =											
1	2	3	4	5	6	7	8	9	10	11	12
6/72	7/72	8/72	9/72	10/72	11/72	6/72	5/72	4/72	3/72	2/72	1/72

c) Write down a table showing conditional probability mass function for X given $Y = 5$.

We take the two numbers in the column for $Y = 5$ *in the joint table, and divide by the probability that* $Y = 5$ *from the marginal table for* Y*. The result is*

X =	
1	2
6/10	4/10

Question 4: A software development firm has two programmers, Alice and Jill, to whom they assign software programming projects at random, with equal probabilities. Alice is a rather good programmer. Half the time, her programs have no bugs, and otherwise they have just a single bug. Jill isn't as good. She has an equal chance of producing a program with no bugs, one bug, two bugs, three bugs, or four bugs (ie, each of these possibilities has probability 1/5). After Alice or Jill finish a program, it is sent to the testing department, where they try to find the bugs. The testers have a 1/3 chance of finding each bug, and whether they find one bug is independent of whether they find any other bug (ie, the events of them finding each bug are mutually independent).

a) What is the expected number of bugs in a program (which might be written by either Alice or Jill) that is sent to the testers?

This question is somewhat analogous to Question 3 — the coin flip there corresponds to the choice of programmer here. We can define the following random variables:

$$A = \begin{cases} 1 & \textit{if Alice wrote the program} \\ 2 & \textit{if Jill wrote the program} \end{cases}$$

$$B = \textit{The number of bugs in the program}$$

The joint probability mass function for A and B can be found with the multiplication rule, using the following facts: $P(A=1) = P(A=2) = 1/2$, $P(B=0\,|\,A=1) = P(B=1\,|\,A=1) = 1/2$, and $P(B=b\,|\,A=2) = 1/5$ for $b = 0,1,2,3,4,5$.

The result is

		B = 0	1	2	3	4
A =	1	1/4	1/4	0	0	0
	2	1/10	1/10	1/10	1/10	1/10

Adding the columns, we can get the marginal probability mass function for B:

B = 0	1	2	3	4
7/20	7/20	2/20	2/20	2/20

From this, we can find the expected value of B from the definition of expectation:

$$E(B) = 0 \times 7/20 + 1 \times 7/20 + 2 \times 2/20 + 3 \times 2/20 + 4 \times 2/20 = 5/4$$

It is also possible to get this answer by averaging the expected number of bugs in a program written by Alice and the expected number of bugs in a program written by Jill, but justifying this mathematically requires material on conditional expectation that we haven't covered yet.

b) What is the expected number of bugs in a program that are not found by the testers?

Intuitively, you might expect that since the testers fail to find 2/3 of the bugs, the expected number of bugs not found will be 2/3 the expected number of bugs computed above — which is $(5/4) \times (2/3) = 5/6$. *And that is indeed correct. But to justify this mathematically requires material on conditional expectation and the binomial distribution that we haven't covered yet.*

We can compute the expected number of bugs not found directly from the definition of expectation if we can find the probability mass function for this random variable (call it N*). We can do this by finding the joint probability mass function for* B *and* N*, and using it to find the marginal probability mass function for* N*. We can find the joint probabilities using the multiplication rule:* $P(B=b, N=n) \;=\; P(B=b)\,P(N=n\,|\,B=b)$*. We get* $P(B=b)$ *from the marginal probability mass function for* B *that we computed above. We get* $P(N=n\,|\,B=b)$ *from the assumption that the tests have a* $1/3$ *chance of finding a bug, independently for each bug present. For example,* $P(N=1\,|\,B=4)$ *can be found by adding together the probabilities for the 4 ways of not finding 1 out of 4 bugs, each of which has probability* $(1/3)^3\,(2/3)^1$*. The result is the following joint probability mass function:*

	N = 0	1	2	3	4
B = 0	7/20	0	0	0	0
1	7/60	14/60	0	0	0
2	1/90	4/90	4/90	0	0
3	1/270	6/270	12/270	8/270	0
4	1/810	8/810	24/810	32/810	16/810

Adding the columns, we can get the marginal probability mass function for N*:*

N = 0	1	2	3	4
0.4827	0.3099	0.1185	0.0691	0.0198

From this, we can find the expected value of N *from the definition of expectation (rounding to four decimal places, and using three decimal places for the final result):*

$$\begin{aligned} E(B) &= 0 \times 0.4827 \;+\; 1 \times 0.3099 \;+\; 2 \times 0.1241 \;+\; 3 \times 0.0691 \;+\; 4 \times 0.0198 \\ &= 0.833 \end{aligned}$$

c) Suppose the testers find exactly one bug in a program. What is the probability that there are no more bugs in this program? What is the probability that this program was written by Alice?

Let F *be the number of bugs found by the testers. Clearly,* $F = B - N$*. We can use this to find the joint probability mass function for* F *and* B *by just rearranging the entries in the table for* N *and* B*. The result is as follows:*

		F	$=$			
		0	*1*	*2*	*3*	*4*
B $=$	*0*	*7/20*	*0*	*0*	*0*	*0*
	1	*14/60*	*7/60*	*0*	*0*	*0*
	2	*4/90*	*4/90*	*1/90*	*0*	*0*
	3	*8/270*	*12/270*	*6/270*	*1/270*	*0*
	4	*16/810*	*32/810*	*24/810*	*8/810*	*1/810*

To find $P(B = 1 \,|\, F = 1)$*, we look at the column for* $F = 1$ *in this table. The answer is*

$$\begin{aligned} P(B = 1 \,|\, F = 1) &= \frac{P(B = 1, F = 1)}{P(F = 1)} \\ &= \frac{7/60}{7/60 \;+\; 4/90 \;+\; 12/270 \;+\; 32/810} \\ &= \frac{7/60}{0.24506} \quad = \quad 0.47607 \end{aligned}$$

The probability that Alice wrote a program in which one bug was found is

$$P(A = 1 \,|\, F = 1) \;=\; \frac{P(A = 1, F = 1)}{P(F = 1)} \;=\; \frac{(1/2)\,(1/2)\,(1/3)}{0.24506} \;=\; 0.34$$