STA 247 — Solution to Assignment #2, Question 1

You roll ten fair six-sided dice. Let the sum of the numbers shown on all ten dice be R. You then flip a fair coin R times. Let the number of times the coin lands heads be H and the number of times the coin lands tails be T. (So $H + T$ will be equal to R.)

Find each of the quantities below. You must produce an actual numerical answer, as a simple fraction (eg, 3/8) or decimal number (eg, 0.375). You must also justify how you obtained your answer in terms of theorems in the book.

a) $E(R)$, the expected value of R.

Solution: We can write $R = R_1 + R_2 + \cdots + R_{10}$, where R_i is the value from the ith roll. From Theorem 2.7-2 or 2.7-3, we can conclude that

$$E(R) \;=\; E(R_1) + E(R_2) + \cdots + E(R_{10}) \;=\; 10\,E(R_i)$$

We can compute $E(R_i)$ (which is the same for all i) as

$$E(R_i) \;=\; (1/6)\,(1+2+3+4+5+6) \;=\; 3.5$$

From which we get that $E(R) = 35$.

b) $\mathrm{VAR}(R)$, the variance of R.

Solution: Again, we write $R = R_1 + R_2 + \cdots + R_{10}$, where R_i is the value from the ith roll. Since the R_i are independent, we can use Theorem 2.7-6 (or 2.7-5) to conclude that

$$\mathrm{VAR}(R) \;=\; \mathrm{VAR}(R_1) + \mathrm{VAR}(R_2) + \cdots + \mathrm{VAR}(R_{10}) \;=\; 10\,\mathrm{VAR}(R_i)$$

We can compute $\mathrm{VAR}(R_i)$ (which is the same for all i) as

$$\begin{aligned}\mathrm{VAR}(R_i) \;&=\; (1/6)\,((1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2) \\ &=\; 2.91666\ldots\end{aligned}$$

From which we get that $\mathrm{VAR}(R) = 29.1666\ldots$

c) $E(H)$, the expected value of H.

First solution: Using Theorem 2.9-1, we can write

$$E(H) \;=\; E(E(H|R))$$

For a given value of R, the number of heads is just the number of "successes" in R independent Bernoulli trials in which a trial is a flip of the coin, with success being a head. The distribution of H given R is therefore binomial with $n = R$ and $p = 1/2$. From Theorem 3.1-2, we know that $E(H|R) = R/2$. We therefore get that

$$E(H) \;=\; E(E(H|R)) \;=\; E(R/2) \;=\; E(R)/2 \;=\; 35/2 \;=\; 17.5$$

Theorem 2.3-2 is used above to go from $E(R/2)$ to $E(R)/2$.

Second solution: By using Theorem 2.3-3 (or 2.7-2, or 2.7-3), we can write $E(R) = E(H + T) = E(H) + E(T)$. The problem is completely symmetrical between H and T, so $E(T) = E(H)$. We also know that $E(R) = 35$. It follows that $E(H) = 35/2 = 17.5$.

d) VAR(H), the variance of H.

Solution: Using Theorem 2.9-2, we can write

$$\text{VAR}(H) \;=\; E(\text{VAR}(H|R)) \;+\; \text{VAR}(E(H|R))$$

As argued above, the distribution of H given R is binomial with $n = R$ and $p = 1/2$. From Theorem 3.1-2, we know that $E(H|R) = R/2$ and $\text{VAR}(H|R) = R/4$. Therefore,

$$\begin{aligned}
\text{VAR}(H) &= E(\text{VAR}(H|R)) \;+\; \text{VAR}(E(H|R)) \\
&= E(R/4) \;+\; \text{VAR}(R/2) \\
&= E(R)/4 \;+\; \text{VAR}(R)/4 \\
&= 35/4 \;+\; 29.1666\ldots/4 \;=\; 16.041666\ldots
\end{aligned}$$

Theorem 2.3-2 is used above to go from $E(R/4)$ to $E(R)/4$ and Theorem 2.4-2 is used to go from VAR($R/2$) to VAR(R)/4.

e) COV(T, H), the covariance of T and H.

First solution: From the discussion just after the definition of covariance on page 90,

$$\text{COV}(T,H) \;=\; E(TH) \;-\; E(T)E(H)$$

We can write $T = R - H$, and therefore $E(TH) = E((R-H)H) = E(RH - H^2) = E(RH) - E(H^2)$ (using Theorem 2.7-2). Using Theorem 2.4-3, we can write $E(H^2) = \text{VAR}(H) + E(H)^2$. Since T and H are completely symmetrical in this problem, we also know that $E(T) = E(H)$. Combining these facts, we get that

$$\begin{aligned}
\text{COV}(T,H) &= E(TH) \;-\; E(T)E(H) \\
&= E(RH) - (\text{VAR}(H) + E(H)^2) - E(H)^2 \\
&= E(RH) - \text{VAR}(H) - 2E(H)^2
\end{aligned}$$

Using Theorem 2.9-1, $E(RH) = E(E(RH|R)) = E(R(E(H|R))$, where the last step is an application of Theorem 2.3-2, considering that R is a constant if we're given the value of R. As argued above, $E(H|R) = R/2$. Therefore $E(RH) = E(R^2/2) = E(R^2)/2$. Using Theorem 2.4-3 again, we write $E(R^2) = \text{VAR}(R) + E(R)^2$, so $E(RH) = (\text{VAR}(R) + E(R)^2)/2$. Putting these facts together, and then substituting known values,

$$\begin{aligned}
\text{COV}(T,H) &= E(RH) - \text{VAR}(H) - 2E(H)^2 \\
&= (\text{VAR}(R) + E(R)^2)/2 - \text{VAR}(H) - 2E(H)^2 \\
&= (29.1666\ldots + 35^2)/2 - 16.041666\ldots - 2 \times 17.5^2 \\
&= -1.458333\ldots
\end{aligned}$$

Second solution: From Theorem 2.8-1, we know that $\text{VAR}(R) = \text{VAR}(T+H) = \text{VAR}(T) + \text{VAR}(H) + 2\text{COV}(T,H)$. From symmetry, $\text{VAR}(T) = \text{VAR}(H)$, so

$$\begin{aligned}
\text{COV}(T,H) &= \text{VAR}(R)/2 - \text{VAR}(H) \\
&= 29.1666\ldots/2 \;-\; 16.041666\ldots \;=\; -1.458333\ldots
\end{aligned}$$

f) $\mathrm{CORR}(T,H)$, the correlation of T and H.

Solution: By definition, $\mathrm{CORR}(T,H) = \mathrm{COV}(T,H) \Big/ \sqrt{\mathrm{VAR}(T)\mathrm{VAR}(H)}$. Since by symmetry, $\mathrm{VAR}(H) = \mathrm{VAR}(T)$, we get that

$$\begin{aligned} \mathrm{CORR}(T,H) &= \mathrm{COV}(T,H) \,/\, \mathrm{VAR}(H) \\ &= -1.458333\ldots/16.041666\ldots = -0.09090909\ldots \end{aligned}$$