You roll ten fair six-sided dice. Let the sum of the numbers shown on all ten dice be $R$. You then flip a fair coin $R$ times. Let the number of times the coin lands heads be $H$ and the number of times the coin lands tails be $T$. (So $H + T$ will be equal to $R$.)

Find each of the quantities below. You must produce an actual numerical answer, as a simple fraction (eg, 3/8) or decimal number (eg, 0.375). You must also justify how you obtained your answer in terms of theorems in the book.

a) $E(R)$, the expected value of $R$.

**Solution:** We can write $R = R_1 + R_2 + \cdots + R_{10}$, where $R_i$ is the value from the $i$th roll. From Theorem 2.7-2 or 2.7-3, we can conclude that

$$E(R) = E(R_1) + E(R_2) + \cdots + E(R_{10}) = 10 E(R_i)$$

We can compute $E(R_i)$ (which is the same for all $i$) as

$$E(R_i) = \left(\frac{1}{6}\right)(1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

From which we get that $E(R) = 35$.

b) $\text{VAR}(R)$, the variance of $R$.

**Solution:** Again, we write $R = R_1 + R_2 + \cdots + R_{10}$, where $R_i$ is the value from the $i$th roll. Since the $R_i$ are independent, we can use Theorem 2.7-6 (or 2.7-5) to conclude that

$$\text{VAR}(R) = \text{VAR}(R_1) + \text{VAR}(R_2) + \cdots + \text{VAR}(R_{10}) = 10 \text{VAR}(R_i)$$

We can compute $\text{VAR}(R_i)$ (which is the same for all $i$) as

$$\text{VAR}(R_i) = \left(\frac{1}{6}\right)((1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2)$$

$$= 2.91666\ldots$$

From which we get that $\text{VAR}(R) = 29.1666\ldots$

c) $E(H)$, the expected value of $H$.

**First solution:** Using Theorem 2.9-1, we can write

$$E(H) = E(E(H|R))$$

For a given value of $R$, the number of heads is just the number of “successes” in $R$ independent Bernoulli trials in which a trial is a flip of the coin, with success being a head. The distribution of $H$ given $R$ is therefore binomial with $n = R$ and $p = 1/2$. From Theorem 3.1-2, we know that $E(H|R) = R/2$. We therefore get that

$$E(H) = E(E(H|R)) = E(R/2) = E(R)/2 = 35/2 = 17.5$$

Theorem 2.3-2 is used above to go from $E(R/2)$ to $E(R)/2$.

**Second solution:** By using Theorem 2.3-3 (or 2.7-2, or 2.7-3), we can write $E(R) = E(H + T) = E(H) + E(T)$. The problem is completely symmetrical between $H$ and $T$, so $E(T) = E(H)$. We also know that $E(R) = 35$. It follows that $E(H) = 35/2 = 17.5$. 

1
d) \( \text{VAR}(H) \), the variance of \( H \).

**Solution:** Using Theorem 2.9-2, we can write

\[
\text{VAR}(H) = E(\text{VAR}(H|R)) + \text{VAR}(E(H|R))
\]

As argued above, the distribution of \( H \) given \( R \) is binomial with \( n = R \) and \( p = 1/2 \). From Theorem 3.1-2, we know that \( E(H|R) = R/2 \) and \( \text{VAR}(H|R) = R/4 \). Therefore,

\[
\text{VAR}(H) = E(\text{VAR}(H|R)) + \text{VAR}(E(H|R)) = E(R/4) + \text{VAR}(R/2) = E(R)/4 + \text{VAR}(R)/4 = 35/4 + 29.1666\ldots/4 = 16.041666\ldots
\]

Theorem 2.3-2 is used above to go from \( E(R/4) \) to \( E(R)/4 \) and Theorem 2.4-2 is used to go from \( \text{VAR}(R/2) \) to \( \text{VAR}(R)/4 \).

e) \( \text{COV}(T, H) \), the covariance of \( T \) and \( H \).

**First solution:** From the discussion just after the definition of covariance on page 90,

\[
\text{COV}(T, H) = E(TH) - E(T)E(H)
\]

We can write \( T = R - H \), and therefore \( E(TH) = E((R - H)H) = E(RH - H^2) = E(RH) - E(H^2) \) (using Theorem 2.7-2). Using Theorem 2.4-3, we can write \( E(H^2) = \text{VAR}(H) + E(H)^2 \). Since \( T \) and \( H \) are completely symmetrical in this problem, we also know that \( E(T) = E(H) \). Combining these facts, we get that

\[
\text{COV}(T, H) = E(TH) - E(T)E(H) = E(RH) - (\text{VAR}(H) + E(H)^2) - E(H)^2 = E(RH) - \text{VAR}(H) - 2E(H)^2
\]

Using Theorem 2.9-1, \( E(RH) = E(E(RH|R)) = E(R(E(H|R))) \), where the last step is an application of Theorem 2.3-2, considering that \( R \) is a constant if we’re given the value of \( R \). As argued above, \( E(H|R) = R/2 \). Therefore \( E(RH) = E(R^2/2) = E(R^2)/2 \). Using Theorem 2.4-3 again, we write \( E(R^2) = \text{VAR}(R) + E(R)^2 \), so \( E(RH) = (\text{VAR}(R) + E(R)^2)/2 \). Putting these facts together, and then substituting known values,

\[
\text{COV}(T, H) = E(RH) - \text{VAR}(H) - 2E(H)^2 = (\text{VAR}(R) + E(R)^2)/2 - \text{VAR}(H) - 2E(H)^2 = (29.1666\ldots + 35^2)/2 - 16.041666\ldots - 2 \times 17.5^2 = -1.458333\ldots
\]

**Second solution:** From Theorem 2.8-1, we know that \( \text{VAR}(R) = \text{VAR}(T + H) = \text{VAR}(T) + \text{VAR}(H) + 2\text{COV}(T, H) \). From symmetry, \( \text{VAR}(T) = \text{VAR}(H) \), so

\[
\text{COV}(T, H) = \text{VAR}(R)/2 - \text{VAR}(H) = 29.1666\ldots/2 - 16.041666\ldots = -1.458333\ldots
\]
f) CORR($T, H$), the correlation of $T$ and $H$.

**Solution:** By definition, CORR($T, H$) = COV($T, H$) / $\sqrt{\text{VAR}(T)\text{VAR}(H)}$. Since by symmetry, $\text{VAR}(H) = \text{VAR}(T)$, we get that

$$\text{CORR}(T, H) = \frac{\text{COV}(T, H)}{\text{VAR}(H)}$$

$$= -1.45833\ldots / 16.04166\ldots = -0.09090909\ldots$$