STA 247, Fall 2003 — Solutions to the Mid-term Test

1. Every year you cook a turkey and a ham for Thanksgiving dinner. You also invite four of your friends over to help eat the turkey and the ham. You have ten friends altogether. Four of your friends like turkey and six of them like ham (none of them likes both turkey and ham). If you select the four friends to invite to dinner at random (with equal probabilities) from among your ten friends, what is the probability that two of the friends you invite will like turkey and two will like ham?

   The sample space consists of all possible ways to choose four friends out of ten. These ways are equally likely, and the number of such ways is

   $$\binom{10}{4} = \frac{10!}{4!\,6!} = 210$$

   The event of interest is the subset of the sample space in which there are two friends who like turkey and two who like ham. Since any two friends who like turkey can go together with any two who like ham, we multiply the number of ways of choosing two friends who like turkey by the number of ways of choosing two friends who like ham to get the total number of points in the event. This is

   $$\binom{4}{2}\binom{6}{2} = \frac{4!}{2!\,2!}\frac{6!}{4!\,2!} = 90$$

   The probability of this event is therefore 90/210 = 3/7.

2. In Toronto, it rains 30% of the time during October. When it is raining, you have a 2% chance of having an accident if you drive to U of T, but when it is not raining, you have only a 1% chance of having an accident. If you drive to U of T at a randomly chosen time in October, what is the probability that you will have an accident?

   Let A be the event of having an accident, and R be the event of it raining. We know that $P(R) = 0.3$, $P(A|R) = 0.02$, and $P(A|R^c) = 0.01$.

   Using third basic axiom of probability (p. 8), we can write the probability of an accident as

   $$P(A) = P(A \cap R) + P(A \cap R^c)$$

   Using the multiplication rule (p. 30), we can get to

   $$P(A) = P(R)P(A|R) + P(R^c)P(A|R^c)$$

   which we could also have gotten to directly with the law of total probability (p. 31). Substituting in the numbers we know, noting that $P(R^c) = 1 - P(R)$, we get

   $$P(A) = 0.3 \times 0.02 + 0.7 \times 0.01 = 0.006 + 0.007 = 0.013$$

3. Recall that three events, A, B, and C, are defined to be mutually independent if and only if all the following are true:

$$\begin{aligned} P(A \cap B \cap C) &= P(A)P(B)P(C) \\ P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \end{aligned}$$

Prove that if A, B, and C are mutually independent, then

$$P(A \cap B^c \cap C) = P(A)P(B^c)P(C)$$

Using the fact that $P(B^c) = 1 - P(B)$, we can rewrite the right hand side as follows:

$$\begin{aligned} P(A)P(B^c)P(C) &= P(A)(1 - P(B))P(C) \\ &= P(A)P(C) - P(A)P(B)P(C) \end{aligned}$$

Using the definition of mutual independence above, we can now write this as

$$P(A)P(B^c)P(C) = P(A \cap C) - P(A \cap B \cap C)$$

Now we use the facts that $A \cap C = (A \cap B \cap C) \cup (A \cap B^c \cap C)$ and that $A \cap B \cap C$ and $A \cap B^c \cap C$ are mutually exclusive. The third basic axiom of probability (p. 8) then allows us to conclude that $P(A \cap C) = P(A \cap B \cap C) + P(A \cap B^c \cap C)$. Combining that with the equation above, we get is what we are trying to prove:

$$P(A)P(B^c)P(C) = P(A \cap B^c \cap C)$$

4. You roll one red die and one green die. Define the random variables X and Y as follows:

$$\begin{aligned} X &= \text{The number showing on the red die} \\ Y &= \text{The number of dice that show the number six} \end{aligned}$$

For example, if the red and green dice show the numbers 6 and 4, then $X = 6$ and $Y = 1$.

Write down a table showing the joint probability mass function for X and Y, find the marginal distribution for Y, and compute $E(Y)$.

Here is a table showing the joint probability mass function, with the marginal distribution for Y on the right:

		$X =$ 1	2	3	4	5	6	
	0	*5/36*	*5/36*	*5/36*	*5/36*	*5/36*	*0*	*25/36*
$Y =$	*1*	*1/36*	*1/36*	*1/36*	*1/36*	*1/36*	*5/36*	*10/36*
	2	*0*	*0*	*0*	*0*	*0*	*1/36*	*1/36*

The expectation of Y is

$$E(Y) = 0 \times 25/36 + 1 \times 10/36 + 2 \times 1/36 = 12/36 = 1/3$$

5. You have been informed that the main U of T web page is accessed an average of 25000 times per day. You have also been told that this web page is accessed 50000 or more times on 1% of the days. Say whatever you can about the standard deviation of the number of accesses in a day.

   Let X be the number of accesses in a day. We know that $\mu = E(X) = 25000$. We also know that $P(X \geq 50000) = 0.01$. Since the number of accesses in a day can't be negative, this is equivalent to $P(|X - 25000| \geq 25000) = 0.01$.

   (Actually, not quite, since the event $|X - 25000| \geq 25000$ includes $X = 0$, which might have non-zero probability — I should have said "more than 50000" rather than "50000 or more" — but no one seems to have been confused by this.)

   Using Chebychev's inequality, we see that

$$0.01 \;=\; P(|X - 25000| \geq 25000) \;=\; P(|X - \mu| \geq 25000) \;\leq\; \sigma^2/25000^2$$

   where σ is the standard deviation of X. From this we get that $\sigma^2 \geq 0.01 \times 25000^2$, and therefore $\sigma \geq 2500$.

   We might be able to say something stronger about σ if we knew more about the distribution, but not if we have to rely only on Chebychev's inequality.