Mean of a Random Variable Times a Constant

If we multiply a random variable by a constant, the mean gets multiplied by this constant too:

$$\mu_{aX} \;=\; a\,\mu_X$$

If X is discrete, we can see this as follows:

$$\begin{aligned}
\mu_{aX} &= \sum_{\substack{\text{values } v \\ \text{of } ax}} v\,P(aX=v) \\
&= \sum_x (ax)\,P(aX=ax) \\
&= a\sum_x x\,P(X=x) \\
&= a\,\mu_X
\end{aligned}$$

Mean of a Sum of Random Variables

The mean of a sum of two random variables is just the sum of their means:

$$\mu_{X+Y} \;=\; \mu_X + \mu_Y$$

When X and Y are discrete, we see this as follows:

$$\begin{aligned}
\mu_{X+Y} &= \sum_x\sum_y (x+y)\; P(X=x \text{ and } Y=y) \\
&= \sum_x\sum_y \; x\; P(X=x \text{ and } Y=y) \\
&\quad + \sum_x\sum_y \; y\; P(X=x \text{ and } Y=y) \\
&= \sum_x x \sum_y P(X=x \text{ and } Y=y) \\
&\quad + \sum_y y \sum_x P(X=x \text{ and } Y=y) \\
&= \sum_x x\,P(X=x) \;+\; \sum_y y\,P(Y=y) \\
&= \mu_X + \mu_Y
\end{aligned}$$

Variance of a Random Variable Times a Constant

If we multiply a random variable by a constant, the variance gets multiplied by this constant *squared*. The standard deviation is multiplied by just the constant:

$$\sigma^2_{aX} \;=\; a^2\,\sigma^2_X, \qquad \sigma_{aX} \;=\; a\sigma_X$$

Why? The variance of aX is the mean of

$$((aX)-\mu_{aX})^2 \;=\; ((aX)-(a\mu_X))^2 \;=\; a^2\,(X-\mu_X)^2$$

The mean of this is a^2 times the mean of $(X-\mu_X)^2$, which is σ^2_X.

We usually use standard deviation when interpreting results, since it scales in the way you would expect when you change units of measurement.

What happens to the variance and standard deviation if we *add* a constant?

Variance of a Sum of Independent Random Variables

The variance of the sum of two *independent* random variables, X and Y, is the sum of the variances:

$$\sigma^2_{X+Y} \;=\; \sigma^2_X + \sigma^2_Y$$

This may not be true if X and Y are dependent.

Proof when $\mu_X = \mu_Y = 0$:

$$\begin{aligned}
\sigma^2_{X+Y} &= \sum_x\sum_y (x+y)^2\,P(X=x \text{ and } Y=y) \\
&= \sum_x\sum_y x^2\,P(X=x \text{ and } Y=y) \\
&\quad + \sum_x\sum_y y^2\,P(X=x \text{ and } Y=y) \\
&\quad + 2\sum_x\sum_y xy\,P(X=x)\,P(Y=y) \\
&= \sigma^2_X + \sigma^2_Y + 2\mu_X\mu_Y \;=\; \sigma^2_X + \sigma^2_Y
\end{aligned}$$

This is one reason we sometimes look at variances rather than standard deviations.

Binomial Random Variables as Sums

If X has the binomial(n,π) distribution, we can express it as

$$X = S_1 + \cdots + S_n$$

where the S_i are independent random variables representing success (1) or failure (0) in each Bernoulli trial.

For all S_i,

$$\begin{aligned} P(S_i = 1) &= \pi \\ P(S_i = 0) &= 1 - \pi \end{aligned}$$

The mean of S_i is

$$1\pi + 0(1-\pi) = \pi$$

The variance of S_i is

$$(1-\pi)^2\pi + (0-\pi)^2(1-\pi) = \pi(1-\pi)$$

The Mean and Variance of a Binomial Distribution

We can figure out the mean and variance of a binomial(n,π) random variable from its representation as

$$X = S_1 + \cdots + S_n$$

The rule for the mean of a sum gives

$$\mu_X = \mu_{S_1} + \cdots + \mu_{S_n} = n\pi$$

The rule for the variance of a sum of independent random variables gives

$$\sigma^2_X = \sigma^2_{S_1} + \cdots + \sigma^2_{S_n} = n\pi(1-\pi)$$

The standard deviation is thus $\sqrt{n\pi(1-\pi)}$.