STA 3000, Problem Set 3. Due in class October 31.

1. Show how the Poisson family of distributions can be expressed in exponential family form, and find natural parameters and natural sufficient statistics for this family. Derive formulas for the mean and variance of a Poisson random variable using Proposition 2.70.

2. Consider the exponential family of distributions for X with density function

$$f_{X|\Pi}(x|\pi) \;=\; c(\pi)h(x)\exp(-\pi^T t(x))$$

   where π is the k-dimensional natural parameter and $t(x)$ is the k-dimensional natural sufficient statistic.

   Prove that $\log(1/c(\pi))$ is a convex function of π — ie, prove that for all π and π',

$$\log(1/c((\pi+\pi')/2)) \;\leq\; (\log(1/c(\pi)) + \log(1/c(\pi')))/2$$

   Hint: Use the inequality of geometric and arithmetic means, which states that $(xy)^{1/2} \leq (x+y)/2$ for all $x, y \geq 0$.

3. Consider a model for data consisting of three observations, $X = (X_1, X_2, X_3)$, which are independent given a value for the parameter $\Theta = (\Theta_1, \Theta_2)$, with $X_1 \sim \text{Poisson}(\theta_1)$, $X_2 \sim \text{Poisson}(\theta_2)$, and $X_3 \sim \text{Poisson}(\theta_1\theta_2)$. Show how to express this model as an exponential family model with the natural sufficient statistic (X_1, X_2, X_3). Show that this model is not minimal (ie, it is degenerate), and reduce it to a model with a two-dimensional sufficient statistic. Show that this sufficient statistic is complete.

4. Consider a model for data consisting of three observations, $X = (X_1, X_2, X_3)$, which are independent given a value for the parameter $\Theta = (\Theta_1, \Theta_2)$, with $X_1 \sim \text{Poisson}(\theta_1)$, $X_2 \sim \text{Poisson}(\theta_2)$, and $X_3 \sim \text{Poisson}(\theta_1 + \theta_2)$. Show how to express this model as an exponential family model with the natural sufficient statistic (X_1, X_2, X_3). Show that this sufficient statistic is not complete. Explain why this does not contradict Theorem 2.74.