Consider the problem of estimating $\theta$ from two data points, $X_1, X_2 \overset{\text{iid}}{\sim} U(\theta, \theta + 1)$. We will look at the following estimator:

$$\delta_0(x) = x_1 - 1/2$$

Consider first squared-error loss, for which $L(\theta, a) = (\theta - a)^2$, with $a$ the real-valued estimate.

a) Find the risk function for $\delta_0$.

b) Apply the Rao-Blackwell theorem to find an estimator $\delta_1$ that should have risk at least as small as $\delta_0$. As the sufficient statistic, use the order statistics, $X_{(1)}, X_{(2)}$.

c) Find the risk function for $\delta_1$.

d) Show that $\delta_1$ is also the Pitman estimator for this problem, by correcting $\delta_0(x)$ by subtracting $E_{\theta=0} [\delta_0(X) | Y = x_1 - x_2]$, where $Y = X_1 - X_2$. Confirm that this is also what you get by finding the mean of the normalized likelihood function.

The Rao-Blackwell theorem applies only to convex loss functions. Consider instead the class, $\mathcal{M}$, of loss functions of the form $L(\theta, a) = f(|\theta - a|)$, with $f$ being a monotonically non-decreasing function (ie, $f(d) \leq f(d')$ if $d \leq d'$).

e) For the specific case of this model, with $\delta_0$ and $\delta_1$ above, prove that for any loss function in the class $\mathcal{M}$, the risk for $\delta_1$ is at least as small as the risk for $\delta_0$.

Suppose that the model is instead that $X_1$ and $X_2$ are IID from a mixture distribution, with probability $9/10$ that $X_i$ is exactly $\theta + 1/2$ and probability $1/10$ that $X_i$ is drawn from the $U(\theta, \theta + 1)$ distribution. In other words, $X_1, X_2 \overset{\text{iid}}{\sim} (9/10)\delta_{\theta+1/2} + (1/10)U(\theta, \theta + 1)$, where $\delta_w$ is a point mass at $w$. We will look at the same estimator, $\delta_0$, as above. If we use the order statistics as the sufficient statistic, the Rao-Blackwell theorem applied to $\delta_0$ will give $\delta_1$ as before.

f) Find a loss function in the class $\mathcal{M}$ for which the estimator $\delta_1$ does not have risk at least as small as $\delta_0$. 