Study question set #2 for STA 3000, Spring 2014

These are for study only, not to hand in for credit. They don’t cover all the topics to be covered on the second test, just mostly those not covered by the assignments.

**Question 1:** Data $X_1, \ldots, X_n \in (0, 1)$ are modelled using a parameter $\theta \in (0, 1)$, with $X_1, \ldots, X_n$ i.i.d. given $\theta$, each with density
\[
f(x|\theta) = \begin{cases} 
(1 - \theta)/2 & \text{if } x \leq 1/2 \\
\theta/2 & \text{if } x > 1/2
\end{cases}
\]
We wish to estimate $\theta$ with squared error loss. The estimator $\delta_0(x) = (1/n) \sum_i x_i$ has been proposed. Using the Rao-Blackwell theorem, find a better estimator.

**Question 2:** Suppose we model binary observations $Y_1, Y_2, \ldots, Y_n$ as being i.i.d. given $\theta \in (0, 1)$, with $P(Y_i = 1|\theta) = \theta$. Suppose that our actual observations are that $y_1 = y_2 = \cdots = y_n = 0$.

A) Suppose we do a frequentist hypothesis test of $H_0: \theta = 1/2$ versus $H_1: \theta \neq 1/2$ in the standard way. What will be the p-value computed from this dataset (as a function of $n$)?

B) Suppose we compare two Bayesian models, differing only in their prior for $\theta$, with the prior for model $H_0$ being that $\theta$ is 1/2 with probability one, and the prior for $\theta$ in model $H_1$ being uniform over (0, 1). Suppose that our prior probabilities for $H_0$ and $H_1$ are equal (both 1/2). What will be the posterior probability of model $H_0$ (as a function of $n$)?

**Question 3:** Suppose we model $Y_1$ and $Y_2$ as being i.i.d. given $\theta \in (0, 1)$, with $P(Y_i = 1|\theta) = \theta$. We first observe $y_1$. If $y_1$ is 0, we stop; otherwise we observe $y_2$ as well.

A) Suppose we estimate $\theta$ by the average of the $y_i$ that we observed (ie, either $y_1$ or $(y_1 + y_2)/2$). Is this estimate unbiased?

B) Suppose we use a uniform prior on (0, 1) for $\theta$, and estimate $\theta$ by its posterior mean given the observed $y_i$. What is the prior expectation of this estimate (ie, the expected value of the estimate averaging over the prior for $\theta$ and the distribution of the observed data)?

**Question 4:** Suppose we model positive real observations $Y_1, Y_2, \ldots, Y_n$ as being i.i.d. given a positive real parameter $\theta$, with each observation having density $f(y|\theta) = \theta \exp(-y\theta)$.

A) What will Jeffreys’ prior be for $\theta$ in this model?

B) Suppose we switch to parameterizing the model by $\mu = 1/\theta$. Derive Jeffreys’ prior in this parameterization, and confirm that it is the same as would be obtained from the prior in (A) after transforming the prior density in the usual way.