Question 1:

Given $\theta \in (1, \infty)$, observations $Y_1, \ldots, Y_n \overset{iid}{\sim} U(0, \theta)$. Suppose that we use as the prior for $\theta$ the distribution with density proportional to $I(\theta > 1)/\theta^2$.

A) Find the posterior density for $\theta$ given $y_1, \ldots, y_n$.

B) Find the predictive density for a new observation, $Y_{n+1}$, given $y_1, \ldots, y_n$. Assume that, given a value for $\theta$, $Y_{n+1}$ is independent of $Y_1, \ldots, Y_n$, and has the same distribution as each of $Y_1, \ldots, Y_n$.

Question 2: Suppose we model binary observations $Y_1, Y_2, \ldots, Y_n$ as being i.i.d. given $\theta \in (0, 1)$, with $P(Y_i = 1 | \theta) = \theta$. Suppose that our actual observations are that $y_1 = y_2 = \cdots = y_n = 0$.

A) Suppose we do a frequentist hypothesis test of $H_0 : \theta = 1/2$ versus $H_1 : \theta \neq 1/2$ in the standard way. What will be the p-value computed from this dataset (as a function of $n$)?

B) Suppose we compare two Bayesian models, differing only in their prior for $\theta$, with the prior for model $H_0$ being that $\theta$ is 1/2 with probability one, and the prior for $\theta$ in model $H_1$ being uniform over $(0, 1)$. Suppose that our prior probabilities for $H_0$ and $H_1$ are equal (both 1/2). What will be the posterior probability of model $H_0$ (as a function of $n$)?

Question 3: Suppose that $X_1, \ldots, X_n | \theta \overset{iid}{\sim} N(\theta, 1)$. Let our prior for $\theta$ be an equal mixture of $N(0, 1)$ and a point mass at zero. In other words, under the prior,

$$P(\theta \leq a) = \begin{cases} \Phi(a)/2 & \text{if } a < 0 \\ 3/4 & \text{if } a = 0 \\ 1/2 + \Phi(a)/2 & \text{if } a > 0 \end{cases}$$

where $\Phi$ is the standard normal CDF. Find the posterior distribution for $\theta$ given $x_1, \ldots, x_n$, expressing it as a Radon-Nykodym derivative with respect to a suitable base measure.