Study Question Set #2 for STA 3000, Fall 2014

These are for study only, not to hand in for credit

Question 1: Consider a model for pairs \((X_1, Y_1), \ldots, (X_n, Y_n)\) in which given a value of \(\theta \in \mathbb{R}\), these pairs are i.i.d. with uniform distribution over the interior of the disk with centre \((\theta, 0)\) and radius one. Find the minimal sufficient statistic for this model, showing that it is minimal sufficient using informal arguments based on either the Bayesian or the classical definition of sufficiency.

Question 2: Suppose we observe \(X_1, \ldots, X_n\), which given a value for a parameter \(\theta \in \Omega\), are independent, with each having \(U(0, \theta)\) distribution.

a) Suppose the parameter space is \(\Omega = (1, \infty)\). Find the minimal sufficient statistic, \(T\), for this model, prove it is minimal sufficient, and describe the conditional distribution for \(X_1, \ldots, X_n\) given \(T\).

b) Suppose the parameter space is \(\Omega = \{1, 2, 3, \ldots\} \) — ie, the positive integers. Find the minimal sufficient statistic, \(T\), for this model, prove it is minimal sufficient, and describe the conditional distribution for \(X_1, \ldots, X_n\) given \(T\).

Question 3: Suppose that we observe pairs \((X_i, Y_i)\) for \(i = 1, \ldots, n\), where each \(X_i\) is a positive real and each \(Y_i\) is a non-negative integer. We model these pairs as being IID from the distribution in which \(X_i \sim \text{Exp}(\theta)\) (ie, from the exponential distribution with mean \(1/\theta\)) and \(Y_i | X_i = x_i \sim \text{Poisson}(\theta x_i)\), where \(\theta\) is a positive real model parameter. In other words, the joint density of an observation \((x, y)\) with respect to a combination of Lesbegue and counting measure is

\[
f(x, y) = \theta e^{-\theta x} \cdot (1/y!) \cdot (\theta x)^y e^{-\theta x}
\]

We are interested in estimating \(\phi = 1/\theta\).

a) Find the minimal sufficient statistic for this model.

b) Suppose we use an improper prior for \(\theta\) that is uniform over \((0, \infty)\). Find the posterior mean and standard deviation of \(\phi\) given \((x_1, y_1), \ldots, (x_n, y_n)\).

c) Show that \((Y_1, \ldots, Y_n)\) is an ancillary statistic for this model.

d) Show that \(\bar{X} = (1/n) \sum_{i=1}^n X_i\) is an unbiased estimator of \(\phi\) (ie, for any \(\theta\), the expectation of \(\bar{X}\) under \(P_\theta\) is equal to \(\phi\)), and find its standard deviation.

e) Find the conditional mean and standard deviation of

\[
\hat{X} = \frac{2 \sum_{i=1}^n X_i}{n + \sum_{i=1}^n Y_i}
\]

given \((Y_1, \ldots, Y_n) = (y_1, \ldots, y_n)\).

f) Show that \(\hat{X}\) is an unbiased estimator of \(\phi\).

g) What are the implications of the above results regarding the desirability of performing inference conditional on an ancillary statistic?
**Question 4:** Given $\theta$, all the random variables $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$, and $Z_1, \ldots, Z_n$ are independent. The distribution of each $X_i$ is Bernoulli($\theta$). The distribution of each $Y_i$ and $Z_i$ is Bernoulli($1/2$). We observe $(U_1, V_1), \ldots, (U_n, V_n)$, where $U_i = X_i + Y_i$ and $V_i = X_i + Z_i$. The $X_i$, $Y_i$, and $Z_i$ are not directly observed.

a) Find a simple form of the minimal sufficient statistic for this model, and prove it is minimal sufficient.

b) Find a non-constant ancillary statistic for this model that is a function of the minimal sufficient statistic, and prove it is ancillary.

c) Comment on the significance of this ancillary statistic for inference (e.g., for the meaning of some confidence interval for $\theta$).