STA 410/2102 — Practice Questions for Second Test

Please note that these questions do **not** cover all the topics that may be on the test.

Question 1: Suppose we numerically evaluate the integral

$$\int_0^1 x^4 dx$$

using the midpoint rule. Using 100 points, the approximation we get is 0.199983333625. Using 1000 points, the approximation we get is 0.199999833333363. The exact answer is of course 1/5. Estimate what approximation we will get if we use the midpoint rule with 2000 points.

Question 2: Suppose we have i.i.d. data points $a_1, \ldots, a_n$ that are measurements of angles in radians, in the range of 0 to 2π. We decide to model this data with a form of the "von Mises" distribution that assigns probability density $K \exp(\cos(a_i - \theta))$ to data point a_i, where $K = 0.1257\ldots$, and θ is an unknown model parameter. Suppose that our prior distribution for θ is uniform over the range 0 to 2π.

Given data $a_1, \ldots, a_n$, we wish to compute the Bayes factor for this von Mises model versus the simple model (with no parameters) that says the a_i are uniformly distributed over the range 0 to 2π. The Bayes factor for model A versus model B is the ratio of the probability of the data under model A to the probability of the data under model B, integrating over the parameters (if any) of each model with respect to the prior.

Write an R function that will compute this Bayes factor using R's `integrate` function.

Question 3: Suppose that we want to obtain points, (x, y), that are uniformly distributed over the diamond shape with vertices at $(0, 1)$, $(-1, 0)$, $(0, -1)$, and $(1, 0)$. Write an R function to do this using Gibbs sampling. You should use $(0, 0)$ as the initial point to start the Markov chain, and then sample alternately for x given y and y given x for 1000 iterations. You should return the pairs of points as a list with elements `x` and `y`, each of which will be vectors 1000 long, containing all the points from the chain (except the initial point).

Question 4: Let the state variable, x, for a Markov chain consist of two components, x_1 and x_2. The possible values for x_1 are 0 and 1. The possible values for x_2 are 0, 1, and 2. (There are therefore six possible values for the entire state: 00, 01, 02, 10, 11, and 12.) Define the distribution π by the probabilities

$$\pi(x) = \begin{cases} 1/4 & \text{if } x_2 = x_1 \\ 1/4 & \text{if } x_2 = x_1 + 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch how the Gibbs sampling procedure would work for this distribution, giving in particular the details of what conditional distributions to sample from when, and what these conditional distributions are.

b) Write down explicitly the transition probabilities for the Gibbs sampling Markov chain that you described above, in which first x_1 and then x_2 are updated. (Ie, write down the 6 by 6 matrix whose entries are $T(x'|x)$ for all x and x'.)

c) Show explicitly from the definition of invariance that the Markov chain you described above leaves π invariant.