STA 410/2102 — Answers to Practice Questions for Second Test

Please note that these questions do **not** cover all the topics that may be on the test.

**Question 1:** Non-negative integer counts $Z_1, \ldots, Z_n$ were generated independently from a Poisson distribution with some unknown mean $\lambda$, which we wish to estimate. Data was collected by writing each $Z_i$ on a separate slip of paper, all of which were put in a box. Unfortunately, only now, when it is time to try to estimate $\lambda$, has it been realized that it's not possible to tell the difference between the number 6 and the number 9, because we don't know which way up each slip of paper is supposed to go. Therefore, all these numbers have been entered into the computer as 6, even though some of them may have actually been 9. Call the data as entered $X_1, \ldots, X_n$. The $X_i$ are independent, but, due the confusion of 6 and 9, the probability distribution for $X_i$ is not Poisson, but is instead:

$$P(X_i = k \,|\, \lambda) \;=\; \begin{cases} e^{-\lambda}\lambda^k/k! & \text{if } k \neq 6 \text{ and } k \neq 9 \\ e^{-\lambda}\lambda^6/6! + e^{-\lambda}\lambda^9/9! & \text{if } k = 6 \\ 0 & \text{if } k = 9 \end{cases}$$

Show how to derive the formulas for an EM algorithm to find the maximum likelihood estimate for $\lambda$ from $X_1, \ldots, X_n$, with the unobserved data being the true counts associated with the values recorded as 6. Write an R function to implement this EM algorithm. It should take as arguments a vector of `x` values, and the number of iterations of EM to do. Use some reasonable initial value for $\lambda$ to begin the EM algorithm.

```
paper_em <- function (x,iters)
{
    lambda <- mean(x)
    for (i in 1:iters) {
        # E step:
        prob_6_is_9 <- (lambda^9/prod(1:9)) /
                           (lambda^6/prod(1:6) + lambda^9/prod(1:9))
        # M step:
        lambda <- (sum(x) + sum(x==6)*(9-6)*prob_6_is_9) / length(x)
    }
    lambda
}
```

**Question 2:** Suppose that we want to obtain points, $(x, y)$, that are uniformly distributed over the diamond shape with vertices at $(0, 1)$, $(-1, 0)$, $(0, -1)$, and $(1, 0)$. Write an R function to do this using Gibbs sampling. You should use $(0, 0)$ as the initial point to start the Markov chain, and then sample alternately for $x$ given $y$ and $y$ given $x$ for 1000 iterations. You should return pairs of points as a list with elements `x` and `y`, each of which will be vectors 1000 long, containing all the points from the chain (except the initial point).

```
diamond_gibbs <- function (iters)
{
    results <- list (x=numeric(iters), y=numeric(iters))
    x <- y <- 0
    for (i in 1:iters) {
        # Sample for x given y
        width <- 1 - abs(y)
        x <- runif(1,-width,+width)
        # Sample for y given x
        width <- 1 - abs(x)
        y <- runif(1,-width,+width)
        # Save state in results
        results$x[i] <- x
        results$y[i] <- y
    }
    results
}
```

**Question 3:** Suppose that we model a single real-valued point, $x$, as coming from a normal distribution with mean $\mu$ and variance one. Suppose also that we use a prior distribution for $\mu$ that has the following density function over the reals:

$$f(\mu) \;=\; (1/2)\exp(-|\mu|)$$

a) Write an R function called `met` that samples from the posterior distribution for $\mu$ using the Metropolis algorithm, with the proposal distribution for $\mu$ being normal with mean equal to the current value and variance one. Your function should take as arguments an initial value for $\mu$ and the number of transitions to do, and return a vector of values for $\mu$ after each transition.

```
met <- function (x,initial,iters)
{
    results <- numeric(iters)
    mu <- initial
    for (i in 1:iters) {
        prop_mu <- rnorm(1,mu,1)
        if (runif(1) < exp ((-abs(prop_mu) + dnorm(x,prop_mu,1,log=TRUE))
                             - (-abs(mu) + dnorm(x,mu,1,log=TRUE))))
            mu <- prop_mu
        results[i] <- mu
    }
    results
}
```

b) Write R commands to use the `met` function from part (a) to find the posterior expected value of $\mu^2$ given the observation $x = 1.5$. Use a starting value of zero for the Markov chain, and assume that the first 100 iterations should be discarded as "burn-in" (not necessarily close to having the desired distribution). You should then estimate the expected value of $\mu^2$ using 1000 iterations after the burn-in iterations.

```
r <- met(1.5,0,1100)
print(mean(r[101:1100]^2))
```

**Question 4:** Let $\pi$ be the distribution on the space $\{1, 2, 3, 4\}$ in which 1 and 2 each have probability 1/3 and 3 and 4 each have probability 1/6. Suppose we define a Markov chain to sample from this distribution using the Metropolis method, with the proposal probabilities being given by

$$g(x^*|x) \;=\; \begin{cases} 1/2 & \text{if } x^* = x+1 \text{ or } x = 4 \text{ and } x^* = 1 \\ 1/2 & \text{if } x^* = x-1 \text{ or } x = 1 \text{ and } x^* = 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Write down the $4 \times 4$ matrix of transition probabilities for the Metropolis method using this proposal distribution and with $\pi$ the distribution that should be left invariant. (The entry in row $i$ column $j$ of the matrix $T$ should be the probability that the next state will be $j$ if the current state is $i$.)

$$\begin{bmatrix} 1/4 & 1/2 & 0 & 1/4 \\ 1/2 & 1/4 & 1/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

b) Suppose that the initial state of the Markov chain is randomly drawn from the uniform distribution on $\{1, 2, 3, 4\}$. What will be the distribution of the next state of the Markov chain after this initial state?

$$[1/4 \;\; 1/4 \;\; 1/4 \;\; 1/4] \;\times\; \begin{bmatrix} 1/4 & 1/2 & 0 & 1/4 \\ 1/2 & 1/4 & 1/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \;=\; [5/16 \;\; 5/16 \;\; 3/16 \;\; 3/16]$$