STA 410/2102, Fall 2015 Assignment #3 — Derivations

For Gibbs sampling, we must find the conditional distributions of every variable given all other variables (and the data). This can be done by writing down the joint probability of everything and then looking at only the factors in this joint probability that involve the particular variable whose conditional distribution is being found. The conditional probability (or density) for this variable is then simply these factors divided by their sum or integral (so that the conditonal probabilities/densities sum/integrate to one).

The joint probability of everything, including the species indicators, z_i, is

$$\prod_{j=1}^{10} P(\mu_j)P(\nu_j)P(\rho_j) \;\cdot\; \prod_{i=1}^{500} P(z_i)P(\log(m_i)|z_i,\mu)P(\log(r_i)|z_i,\nu)P(s_i|z_i,\rho)$$

$$= \prod_{j=1}^{10} N(\mu_j;1,2^2)N(\mu_j;1,2^2) \;\cdot\; \prod_{i=1}^{500} \alpha_{z_i} N(\log(m_i);\mu_{z_i},0.08^2)N(\log(r_i);\nu_{z_i},0.10^2)\rho_{z_i}^{s_i}(1-\rho_{z_i})^{1-s_i}$$

To sample for one of the z_i, we isolate just the factors involving it:

$$\alpha_{z_i} N(\log(m_i);\mu_{z_i},0.08^2)N(\log(m_i);\mu_{z_i},0.08^2)\rho_{z_i}^{s_i}(1-\rho_{z_i})^{1-s_i}$$

This gives conditional probabilities that are the same as were used in the E step in Assignment 2.

To sample for one of the μ_j, we look at the factors containing it, which are:

$$N(\mu_j;1,2^2)\prod_{i=1}^{500} N(\log(m_i);\mu_j,0.08^2)^{I(z_i=j)}$$

where $I(z_i=j)$ is one if $z_i=j$ and zero otherwise. This gives as the conditional distribution for μ_j the normal distribution with precision (ie, 1/variance) of $\tau = 1/2^2 + \sum_i I(z_i=j)/0.08^2$ and mean $(1/2^2 + \sum_i I(z_i=j)\log(m_i)/0.08^2)\,/\,\tau$. (See Example 1.2 in the textbook.)

Sampling for $= nu_j$ is entirely analogous to sampling from μ_j.

To sample for ρ_j, we isolate the following factors from the joint probabilty:

$$\prod_{i=1}^{500} \rho_{z_i}^{s_i I(z_i=j)}(1-\rho_{z_i})^{(1-s_i)I(z_i=j)}$$

This has the form of a Beta distribution with parameters $1+\sum_i s_i I(z_i=j)$ and $1+\sum_i(1-s_i)I(z_i=j)$.